Systems Analysis for Sustainable Development

Spring 2013

Lecture 3, Wednesday 23 Jan

Hans Liljenström

Dept. of Energy and Technology, SLU

hans.liljenstrom@slu.se

Today's session

Lecture 3/4

- Modeling principles
- Stucture and behaviour
- System qualities:
  - Stability and equilibrium
  - Response to a disturbance
  - Sensitivity to changes
  - Controllability
  - Observability
- Population dynamics and evolutionary strategies

Different kinds of systems/models

- Deterministic or stochastic systems
- Static or dynamic systems
- Linear or non-linear systems

(The mathematical methods for linear systems are quite clear. For non-linear systems, mathematical analysis usually fails, and computer simulations are called for).

Different types of models

	Static	Dynamic
Stochastic	*Statistical models*	***Dynamic & stochastic models***
Deterministic	*Algebraic models*	*Differential equ. models*

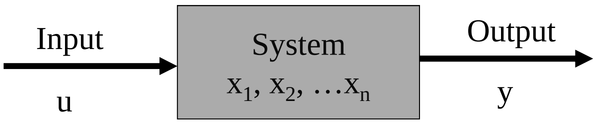
Many ecological, biological, agricultural, medical etc. systems belong here!

Structure and behaviour

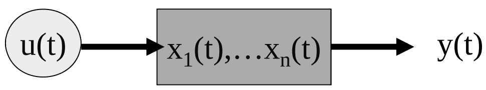
- **Static systems**: Changes can only come from external input
- **Dynamic systems**: Changes are generated by the system itself.
  - Feedback (positive or negative)
  - Implementation of a purpose (control)

Static and dynamical systems

A static system, without time dependence, is described by:

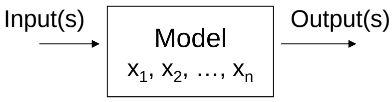


A continuous, dynamical system, is described by:



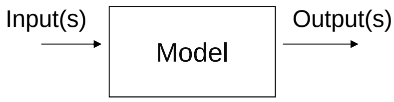
Internal or external description

Internal model
(structural model)



Description of the internal structure. How the components affect each other.

External model
(”black box”)

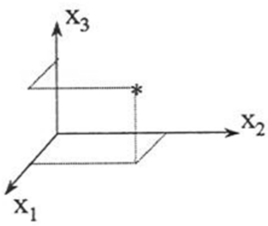


Only input-output description

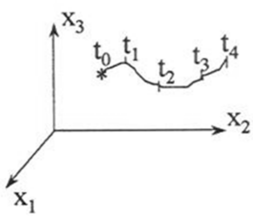
Internal description and state space

A system is described by an internal model with the states, $x_1, x_2, …., x_n$.

At every point in time, these states have certain values, e.g. $x_1 = 2.7$, $x_2 = 7.3$, and $x_3 = 3.4$. Hence, the state at each point in time can be represented by a point in an n-dimensional state space.



When the states change over time, the point forms a trajectory in state space



Modelling dynamical systems

Often, a dynamical system can be modelled with differential equations of the type:

$$\frac{dx}{dt} = f(x)$$

Such equations describing the dynamics of a model are referred to as ***state equations***.

Dynamical systems

Four main mathematical ideas have been developed to characterize time series:

- steady states (equilibrium, fixed point)
- oscillations (limit cycle)
- chaos (strange attractor)
- noise

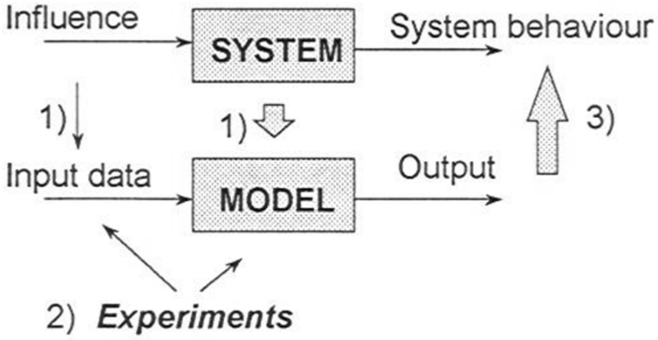
Analytical or numerical solutions

Given a system of state equations our task is to calculate how the quantities will change over time, i.e. to solve the equation system.

In principle, there are two types of solutions:

1. Analytical (provides good insight into the problem, but is only possible in very special cases)
2. Numerical simulation (can handle all types of models, even complex, but is dependent on a set of initial values and model parameters, as one *experiment*).

Simulation = model experiment



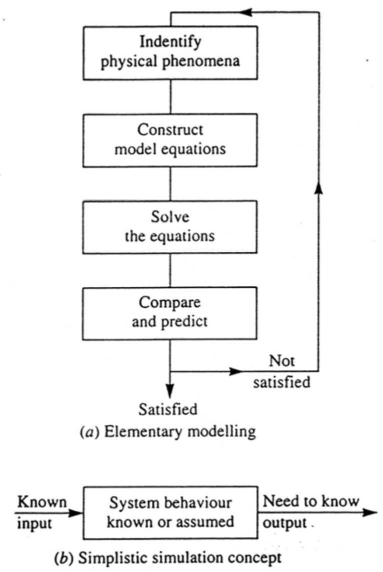
Mathematical model building

specific	⟷	general
model estimation	⟷	first-principle models
numerical	⟷	analytical
stochastic	⟷	deterministic
microscopic	⟷	macroscopic
discrete	⟷	continuous
qualitative	⟷	quantitative

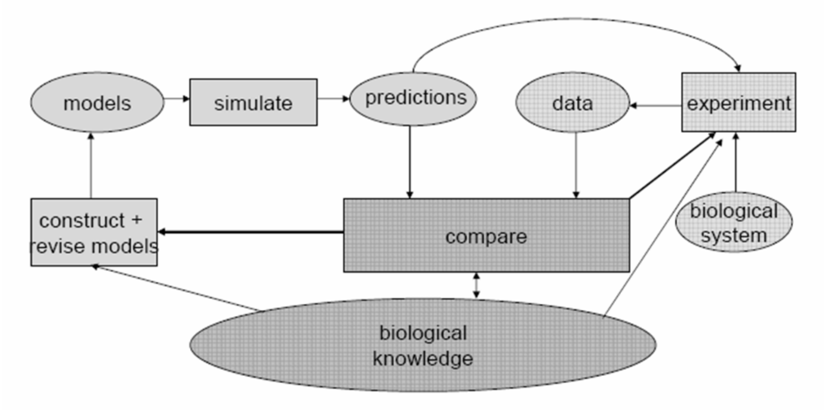
A *correct* model does not exist!

- A model is good if it describes the relevant aspects of the system according to the purpose
- A model/theory is better than another if it:
  - describes more
  - is better tested
  - is ”smaller” (more comprehensive)

Modelling strategies



An Integrated Modeling Process



The modelling process

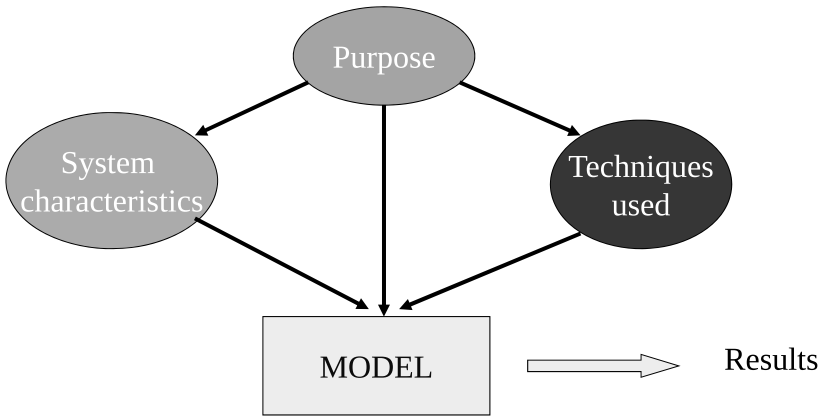
- Which components should be included in the model?
- Which components affect which?
- Are these ”causal arrows” positive or negative?
- Formulate the relations in terms of equations.
- Assemble the equations into an equation system
- Solve the equation system => the behaviour

Schemes

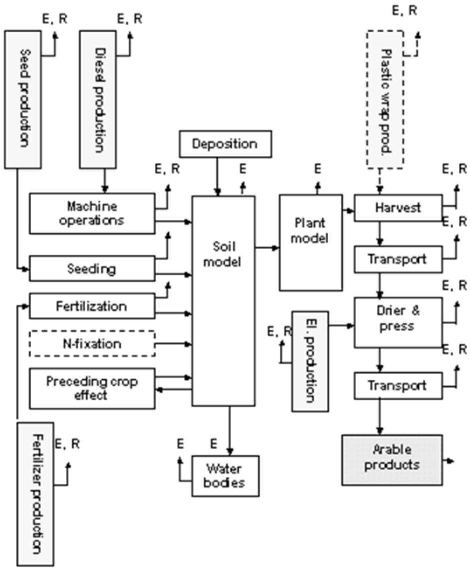
Math or comp. model

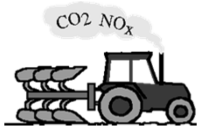
Analytical sol. Or simulation

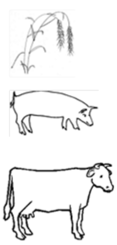
What affects the model?



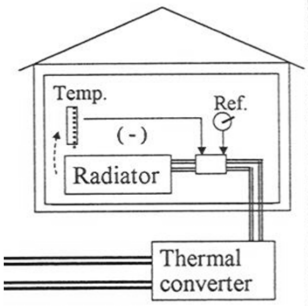
Example: A farm







Feed-back and feed-forward



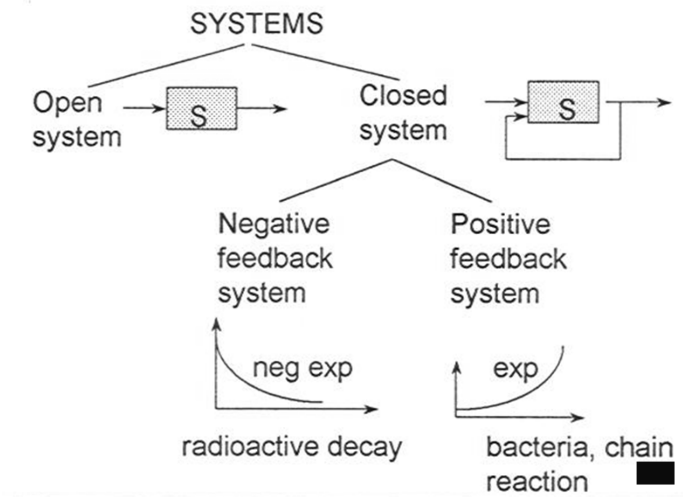
Feedback control

Feedback & feed-forward control

Feedback adjusts to the reference *after* a disturbance

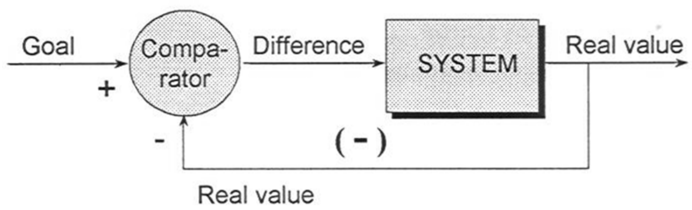
Feed-forward is related to planning in order to react *in advance*

Dynamic systems behaviour



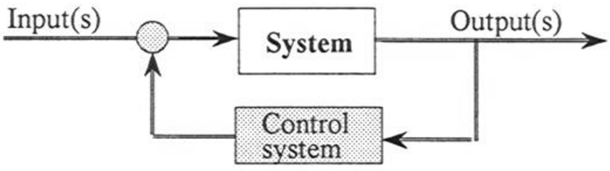
Control - The purpose implemented in a closed loop

A negative feedback mechanism enables control by comparing the ”real value” to the ”goal”.



This is the universal mechanism for control in nature, organisms, society, technical systems, etc.

Feedback in automatic control

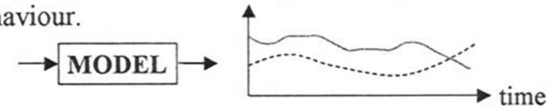


With feedback, a system may be changed in many ways. This is treated in *automatic control* (”Reglerteknik”), and can be used to:

• obtain a goal/purpose
• stabilise an unstable system
• increase or decrease the sensitivity to external influences
• increase or decrease the rates in a system
• improve system properties in many other ways…

System Qualities

To present all possible outcomes of a system/model is not a practical way. We need some concepts to comprehend the system behaviour.

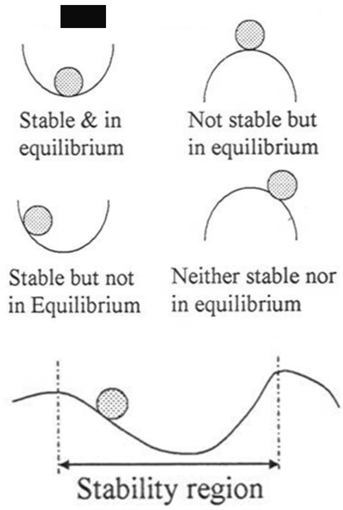


1. Stability and equilibrium
2. Response to a disturbance
3. Sensitivity to changes
4. Controllability
5. Observability

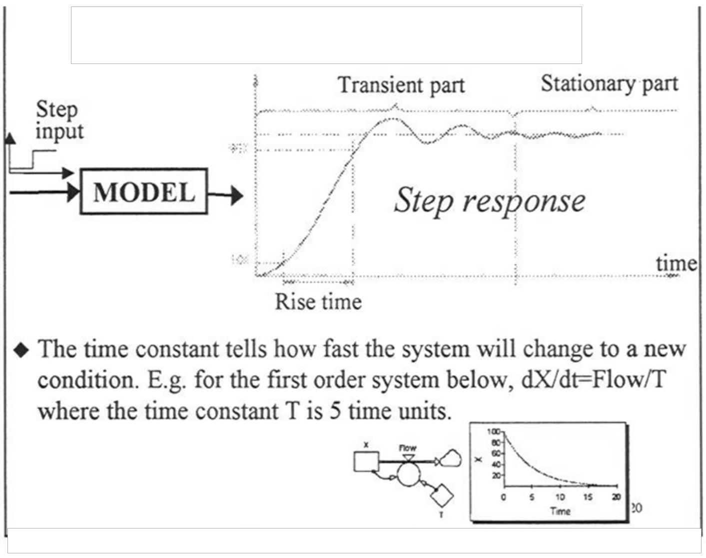
These system properties can be modified with a feedback mechanism.

1. Stability and equilibrium

- Stability tells how a system reacts on (small) disturbances. If it recovers, it is stable (resilient) to that disturbance
- Equilibrium is a matter of balance

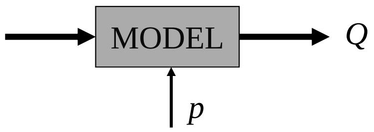


2. Response to a disturbance



3. Sensitivity

How much does a quantity, Q, change when a parameter, p, in the system/model is changed?



This parameter sensitivity is quantified by dQ/dp (absolute changes), or $(dQ/Q)/(dp/p)$ (relative changes, e.g. given in per cent).

There is often a ”stability-sensitivity dilemma”

The stability – sensitivity dilemma

How can a system be sensitive and adaptive, while maintaining a sufficient stability?

It is essential that the system is stable to short-term fluctuations, or common insignificant events, while it should also be able to react to weak signals and rare important events, as well as adapting to long-term changes.

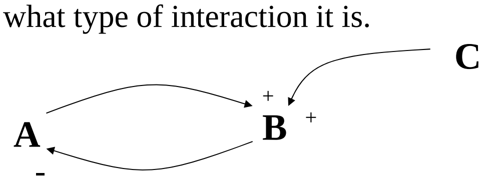
How to achieve this?

4/5. Controllability & observability

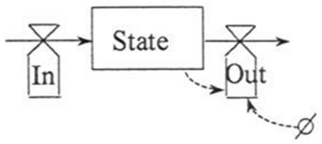
- Controllability: To what degree can you control a system, so that its states take on some predetermined values (e.g. number of rabbits, pH value, amount of wood in forest etc.)?
- Observability: Can you observe or deduce the values of the states?
- Some parts of a system may be controllable, others not. Some states may be observable, others not

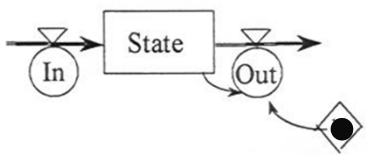
Schemes to describe a system

Causal scheme: Describes how the system components are structurally related, which component interacts with which, and what type of interaction it is.

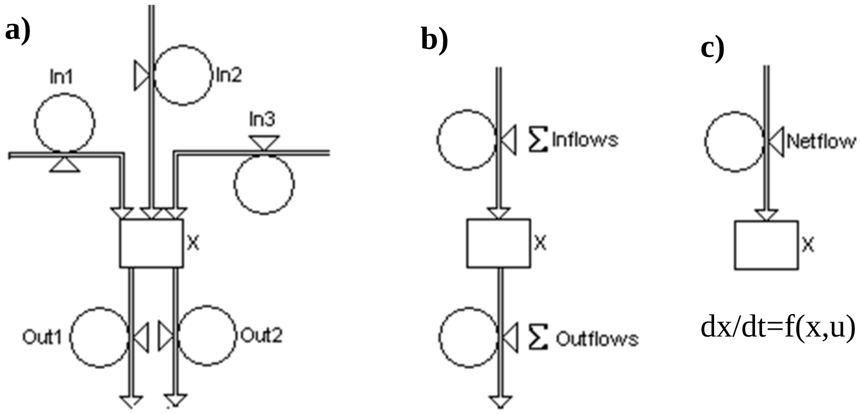


Dynamic scheme: For example, DYNAMO or Powersim.
Dynamics is created when a state is changed due to in- and outflows





1. FLOW MODELS (e.g. Powersim)



The state only changes because of in- and out-flows. **a)** This may be shown explicitly for each flow. **b)** We can sum up the flows to one total inflow and one total outflow. **c)** We can collect all flows into one Netflow $f_{Net}=\sum$Inflows - $\sum$Outflows. Then we can write $dx/dt=f_{Net}(x,u)$ which is what we will use int the mathematical discussions below.

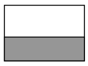
State - the dynamic equation

- The only dynamic component is the state $\boxed{x}$
- The state is dynamically related to the flows via a differential (or integral) equation: dx/dt=f(x,t).
- The solution is achieved by integrating over time:

$$x(t) = x(0) + \int_0^t f(x,t)dt$$ where x(0) is the initial value.

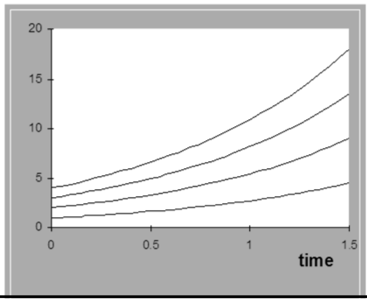
- In simulation the integration is done numerically.

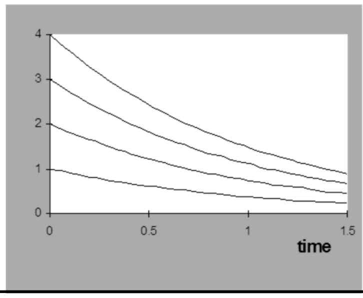
Algebraic equations

- Initial value 

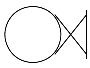
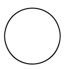
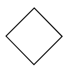
dx/dt=f(x,t) describes how x changes over time.

But not where x is. By defining $x(t=0)=x_0$ we have picked a start value which together with the "change equation" dx/dt=f(x,t) defines the development over time.



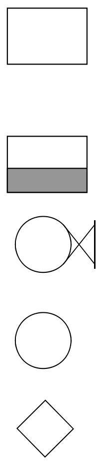


Algebraic equations - continued

- Rate equation 
- Auxiliary equation 
- Constant equation 

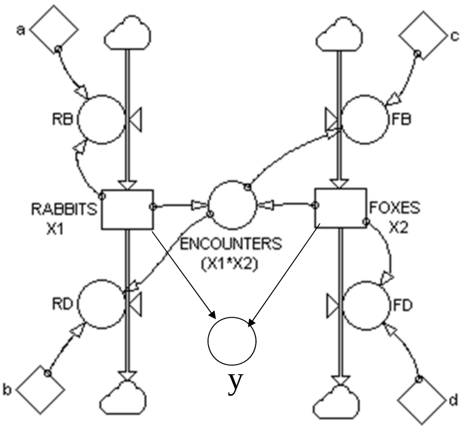
Only **+ - *** and ***/*** are involved in an algebraic relation (no derivatives).

The building blocks in Powersim



Dynamic component

Algebraic components



$$\begin{cases} dx1/dt = a \cdot x1 - b \cdot x1 \cdot x2 \\ dx2/dt = c\, x \cdot 1 \cdot x2 - d \cdot x2 \\ y = x1 + x2 \quad \text{(total number of animals)} \end{cases}$$

(y is the output we want to observe!)

2. THE MODEL IN MATHEMATICAL FORM

A simple system with one input and one output

Structural model

$$\begin{cases} dx_1/dt = f_1(x_1, x_2, x_3, u) \\ dx_2/dt = f_2(x_1, x_2, x_3, u) \\ dx_3/dt = f_3(x_1, x_2, x_3, u) \\ y = g(x_1, x_2, x_3, u) \end{cases}$$

Black box model

$d^3y/dt^3+a_1d^2y/dt^2+a_2dy/dt+a_3 y=u$

(Information about inners structure is lost.)

An n:th order differential equation can always be rewritten as *a set of n first order differential equations*!!

Note the difference between a physical outflow (Powersim) and output y! The output y is what you look for - it does not have to be a flow!

3. ANALYTICAL SOLUTION OR SIMULATION?

Given a system of differential and algebraic equations (including initial values for each state) the task is to calculate how the quantities change over time.

- In mathematics there are methods to do this if the equations are simple enough.
- Numerical calculations begin at the starting time. Then the changes caused by the differential equations are calculated a small step of time ahead and the algebraic equations are recalculated (because the states have changed). Stepwise new approximations are calculated time step by time step until you finish the calculations. This works for any complex model.

Analysis and simulation

Analytical solution

Advantage**:** The solution is given in a closed form (*formula*), which *provides good insight into the problem.* You get all solutions for varying initial values and model parameters. E.g. the system dx/dt=-ax and x(0)=b gives the solution $x(t)=b\cdot e^{-at}$.

Drawback: It is *only possibly in very special cases* to find analytical solutions, e. g. for linear or very simple non-linear models. In some cases approximate solutions can be found by linearisation of a non-linear model.

Simulation

Advantage: Can handle *all types of models*. Complexity is not the problem.

Drawback: *One simulation is just an experiment*. To get a good understanding of the model it is often necessary to make a large number of simulations. Still the results cannot be summarised in a formula.

Population dynamics

For the continuous change of population size, we have:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where

N = number of individuals in the population at time t

r = maximum rate of growth

K = carrying capacity of population, number of individuals at equilibrium

Exponential growth

linear

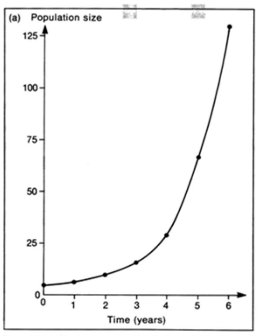


Figure 5.1a The increase in size of a hypothetical population through time using the data in Table 5.1. The population shows exponential growth.

semi-log

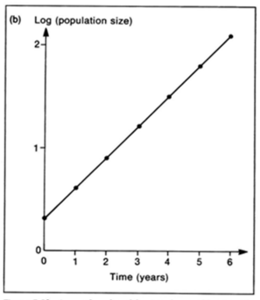
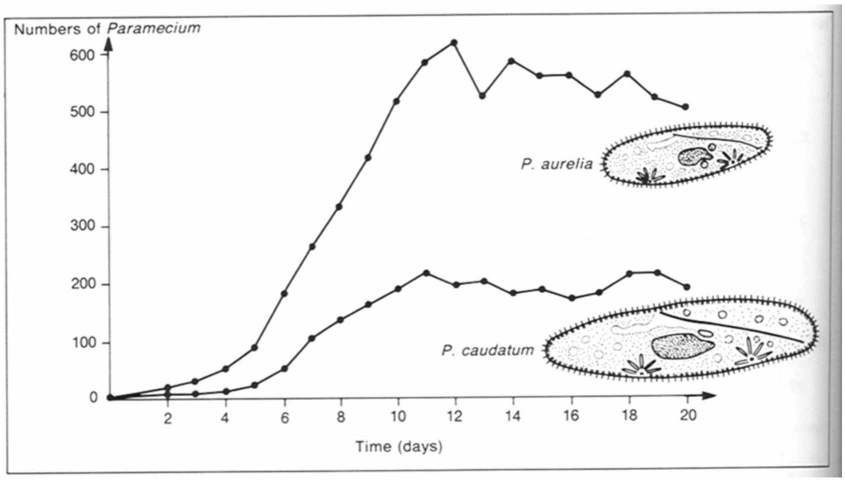
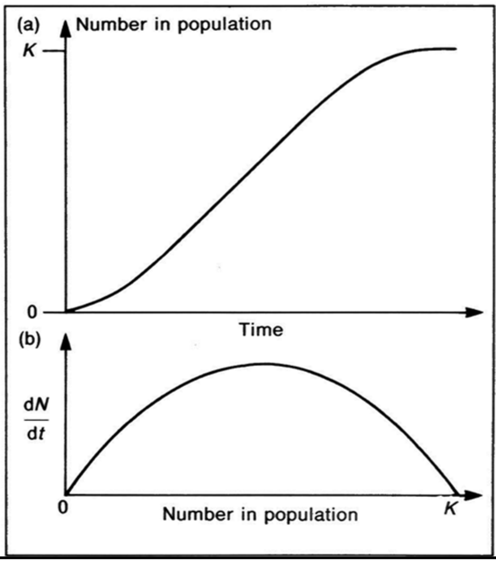


Figure 5.1b A semi-log plot of the data from Table 5.1 with the population size plotted as $\log_{10}$ population size.

Population growth curves



Population change under logistic growth



Evolutionary strategies

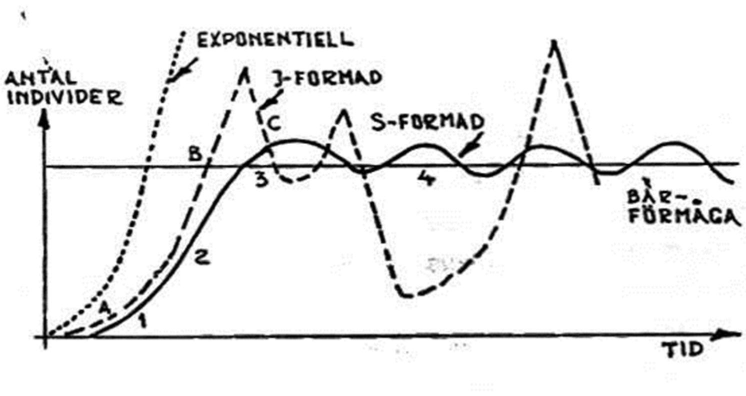
Growth curves can typically be *J-shaped* or *S-shaped*, depending on the *life strategies* of the species.

The growth process can be dividet into four phases:

- phase 1, establishment, slow growth
- phase 2, maximal growth
- phase 3, the growth is leveling off as the carrying capacity of the environmant is approached; regulating processes
- phase 4, dynamic equilibrium (for S-shaped growth)

Organisms with *J* growth have an ”*r-strategy*”, organisms with *S* growth have a ”*K-strategy*”.

Evolutionary strategies



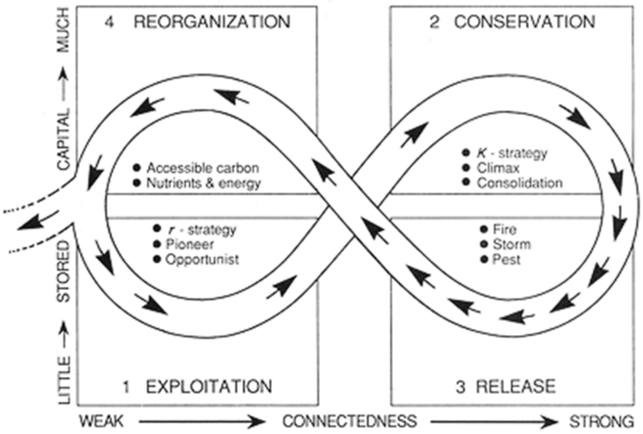
Characteristics of populations at the extremes of r- and K-strategy

Character	Continuum	
	r-strategy	*K*-strategy
Population size	Variable Usually below the maximum the environment can support	Constant In equilibrium Near the maximum the environment can support
	Emigration common Recolonisation high	Recolonisation uncommon
Mortality	Often high Variable Not density dependent*	Often regular Density dependent*
Survivorship curve (semi-log plot)		
Competition	Poor competitor	Good competitor
Life span	Short	Usually more than one year to very long
Environment	Variable and unpredictable	Constant or variable but predictable

Regulation of population dynamics

- Types of regulation (environmental factors)
  - Constantly limiting factors
  - Variably limiting factors
  - Unpredictable factors
- Space
- Food and water
- Territories
- Herbivores and predators
- Weather and climate
- Parasites and diseases
- Natural disasters
- Self-regulation and stress

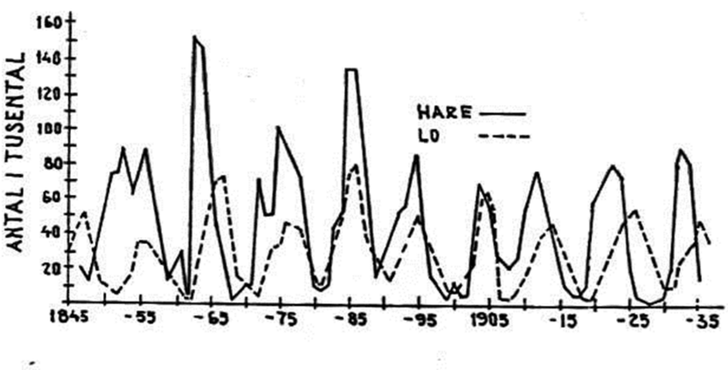
The Adaptive Cycle



The model of the adaptive cycle focuses attention upon **processes of destruction and reorganization**, which are often neglected in favor of growth and conservation. Including these processes provides a more complete view of system dynamics that **links together system organization, resilience, and dynamics.**

(Holling, 1992)

Dynamical balance between predator and prey



The Lotka-Volterra model

The Austrian mathematician A.J. Lotka (1880-1949) and the Italian mathematician V. Volterra (1860-1940) suggested a simple model for the way populations of predators and prey interact.

Let $x_1 = f(t)$ denote the predator population and $x_2 = g(t)$ denote the prey population at time *t*. The predator could be a foxes, and the prey could be a rabbits.

Lotka-Volterra model (contd.)

Assume that, in the absence of predators, so would the prey population *increase* with a rate of px_1 proportional to the amount, but in the presence of predators, these would consume the prey with a rate of qx_1x_2 proportional to the product of the numbers of predators and prey (with p and q positive constants).

The netto change, dx_2/dt of x_2 is given by the rate of growth of the prey, minus the rate of change at which the prey is eaten by the predator, i.e.

$$\frac{dx_2}{dt} = px_2 - qx_1x_2 \qquad (1)$$

Lotka-Volterra model (contd.)

Assume that, if there were no prey, the predators would starve and the population would *decrease* with a rate of rx_1 proportional to the number, but in the presence of prey, the predator population would increas with a rate, sx_1x_2 (with r and s positive constants). These assumptions result in a second DE,

$$\frac{dx_1}{dt} = -rx_1 + sx_1x_2 \qquad (2)$$

The equations (1) and (2) make up a system of DE, called the *Lotka-Volterra model*.

The Lotka-Volterra model

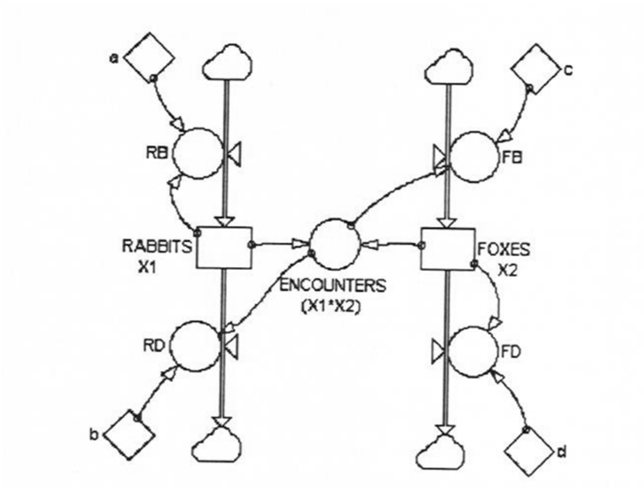
For two species that som interact in a predator-prey relation, we can set up a Lotka-Volterra model, where x_1 represents the prey and x_2 the predator:

$$\frac{dx_1}{dt} = x_1 - \alpha x_1 x_2$$

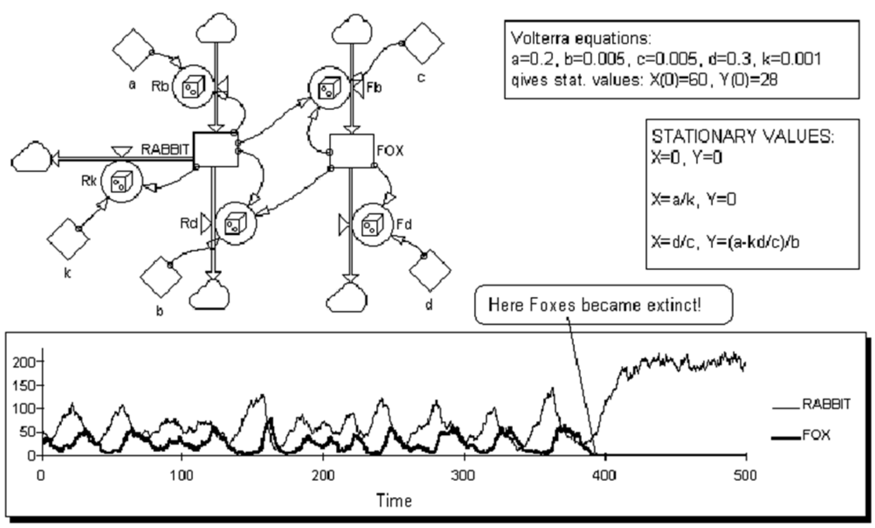
$$\frac{dx_2}{dt} = -x_2 + \beta x_1 x_2$$

Lotka-Volterra model

In Powersim denotation:



<u>**Lotka-Volterra model**</u>



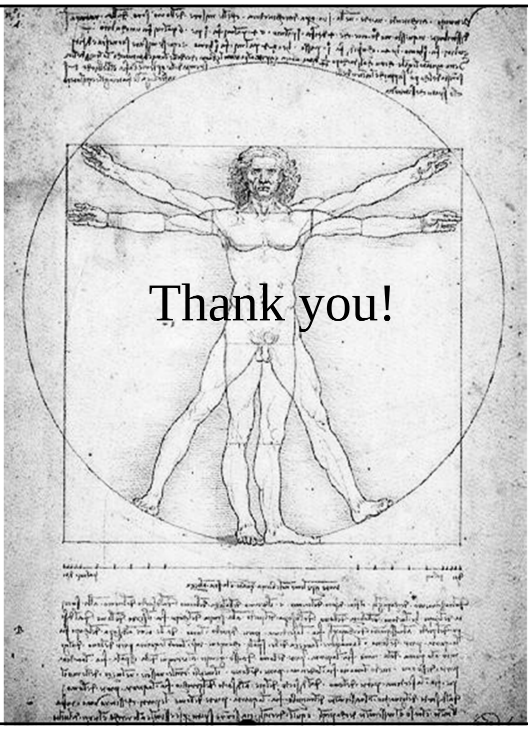
Conclusions: 1) Stochastics excites dynamics! (Started in equilibrium!)
2) A stochastic model may switch to another dynamic ***mode***!

Why Poisson simulation?

Stochastics excites dynamics and dynamics change the stochastic conditions

- If modeled separately, both the statistic and the dynamic estimates will be wrong!
- Average from stochastic model may differ from that of a deterministic model.
- The model may switch between modes (cfr. Volterra).
- Deterministic models that behave exactly the same may can behave quite differently when stochastics are added.
- Adds statistical estimates to a dynamic model.

Dynamics and stochastics must be treated together when both aspects are important !!!



Thank you!